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LETTER TO THE EDITOR

$z^2 - 11z - 1$ as an algebraic invariant for the hard-hexagon model

D W Wood and R W Turnbull

Mathematics Department, University of Nottingham, University Park, Nottingham NG7 2RD, UK

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Abstract. The limiting partition function per site $\Xi_\infty(z)$ for the hard-hexagon model has two singular points at the roots of $z^2 - 11z - 1 = 0$. These two points appear to be algebraic invariants associated with algebraic functions defined on an infinite sequence of finite transfer matrices.

The hard-hexagon lattice gas model (Baxter 1980, 1982) is one of a very small number of exactly solvable lattice models in statistical mechanics. The model which is defined on a triangular lattice of $N = 3n \times m$ sites is exactly solvable only in the sense that the grand canonical partition function per site $\Xi_\infty(z)$ and the order parameter per site $R_\infty(z)$ can be obtained in the double limit of both m and $n \to \infty$. That is to say that $\Xi_\infty(z)$ or $R_\infty(z)$ cannot be explicitly obtained through a limiting process performed on exactly known functions $\Xi_N(z)$ or $R_N(z)$ with either N finite or n finite and $m \to \infty$ (semi-infinite lattice strips). The only example in lattice statistics where such a process is possible remains the Onsager-Kaufman solution of the two-dimensional Ising model, where the partition function Z_{mn} on an $m \times n$ square lattice can be obtained explicitly (Kaufman 1949). The intractible 'analysis' of the full limit of $m, n \to \infty$ is not present for a sequence on n of $n \times \infty$ systems. Here the mathematical aspects are purely algebraic in character and well defined in terms of algebraic functions. The connection between algebraic functions and the limiting emergence of critical point behaviour in lattice models has recently been explored by Wood (1987, 1988) and Wood *et al* (1987). From the algebraic point of view the partition function per site, in the present case Ξ_∞, is the limit function of an infinite sequence of algebraic functions, and the critical point, or any other non-physical singularities, in the limiting partition function per site are limit points in sequences of algebraic branch points. It is also possible to construct sequences of algebraic numbers (roots of one-variable polynomial equations) which converge to the critical point. It is of course an open question under what circumstances the limit functions and limit points remain algebraic functions and algebraic numbers, respectively. In the present case Baxter's solution (Baxter 1980, 1982) shows that the two singular points in the activity plane are algebraic and given by the roots of the quadratic

$$z^2 - 11z - 1 = 0. \tag{1}$$

In Baxter's original solution Ξ_∞ and R_∞ were both obtained in parametric form where the explicit dependence upon z was not known. However, in a recent remarkable paper Joyce (1988) has shown that both $\Xi_\infty(z)$ and $R_\infty(z)$ are indeed algebraic functions

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and thus the critical point behaviour is entirely represented by algebraic branch points and algebraic cycles. The hard-hexagon model, in fact, forms the second example of this mechanism since in earlier work Joyce (1975a, b) has shown that the thermodynamic limiting functions for the three-spin Ising model are also algebraic functions (see also Wood 1987).

Given that $\Xi_\infty(z)$ is both an algebraic function and the limit function of a sequence of algebraic functions, are there any algebraic invariants exhibited by members of this sequence? The purpose of this letter is to report that the two roots of (1) (the singular points of Ξ_∞) are invariant singular points on an infinite sequence of algebraic functions which are generated from an irreducible factor of the characteristic equations of the sequence of finite transfer matrices, $T_{3n}(z)$ for $3n\times\infty$ lattice strips. This is a 'computer discovery': an algebraic proof eludes us, but were one to follow it would probably have something to say about the unknown function sequence $\Xi_N(z)$. Algebraic invariants of this type are usually associated with a simple duality symmetry and previous examples have been given by Wood *et al* (1987). It is interesting that this is *not* the case here.

Consider a general $n\times\infty$ lattice strip with cyclic boundary conditions imposed in all dimensions. Let the transfer matrix be $T_n(z)$, where z is some suitable variable for the problem; in a Potts model, for example, $z=\mathrm{e}^K$, and in the hard-hexagon model z is the activity. The partition function per site is $\Lambda_0^{1/n}$, where Λ_0 is a branch of an algebraic function

$$F_n(\Lambda, z)=\sum_{\alpha\beta} A_{\alpha\beta}\Lambda^\alpha z^\beta=0 \qquad (A_{\alpha\beta}\text{ an integer}) \tag{2}$$

which is an irreducible polynomial factor of the full characteristic equation of $T_n(z)$. Λ_0 is identified as the branch of maximum modulus for real positive z (see Wood 1987, 1988). The resolvent function $R_n(h, z)$ is defined by eliminating Λ between the polynomial (2) and the polynomial equation $F(h\Lambda, z)=0$. Thus, for a given value of z the root set $\{h_i\}$ of the polynomial equation

$$R_n(h, z)=\sum_{\alpha\beta} B_{\alpha\beta}z^\alpha h^\beta=0 \qquad (B_{\alpha\beta}\text{ an integer}) \tag{3}$$

contains all the pair ratios of the root set $\{\Lambda_k\}$ in (2). Technically the root set $\{h_i\}$ should be reduced by eliminating all the trivial roots at $h=1$ (excluding the branch points of Λ). If h is in $\{h_i\}$ then so is h^{-1}. The polynomial equation (3) defines an algebraic function $z(h)$ and generates an algebraic curve via the branches $\{z_j(h)\}_n$.

The domain of interest for our problem is $|h|=1$ and h real. A subset of the curves $\{z_j(h)\}_n$ on $|h|=1$ are the locus of points in the z plane where the root set $\{\Lambda_k\}$ contains pairs which are simultaneously maximum and equal in modulus. This has been denoted by C_n^{1+} (Wood 1987, 1988); the endpoints of C_n^{1+} are branch points of Λ in (2). In the limit of $n\to\infty$ these branch points converge onto the singular points of the partition function per site. We can extend C_n^{1+} through its branch point ends by adding to $|h|=1$, h real. In this way complex conjugate components of C_n^{1+} can be made to intersect the real z axis. One of these intersection points converges to the real critical point in the limit of $n\to\infty$. Such an intersection point for n finite is necessarily a branch point of the set $\{z_j(h)\}_n$ where (3) has a *double* root. If we define $\Delta_n(h)$ to be the discriminant of (3), then one of the roots of $\Delta_n(h)$, h_0 say, corresponds to our real axis intersection point and (3) with $h=h_0$ has a root which converges to the true critical point. Within this panoply of algebraic functions, algebraic invariants for models with a simple self-duality symmetry can arise very simply. Let u be the duality variable (for example $u=\sinh 2K$ for the conventional Ising model or $u=(e^K-1)/\sqrt{q}$ for the q-state Potts model). The effect of self-duality on (2) is simply that the equation can always be written in the form

$$\sum_\alpha \phi_\alpha(w)y^\alpha = 0 \tag{4}$$

where $\Lambda = g(u)y$, $w=u+u^{-1}$ and where $g(u)$ is some trivial multiplicative factor. Since on the circle $|u|=1$ the roots are real or in complex conjugate pairs, the domain $|h|=1$ and real h always generates $|u|=1$ as a member of the algebraic curves $\{z_j(h)\}_n$. Thus all of the circle $|u|=1$, including the real intersection points, are invariant to n, $u=1$ being the real critical point.

For the hard-hexagon model, although the algebraic curves $\{z_j(h)\}_{3n}$ do not contain any invariant arc lengths, the two roots of (1) always exist as intersection points with the real axis. This means that, for any $n \geqslant 2$, the discriminant $\Delta_{3n}(h)$ contains at least two roots, h_0 and h_0', say, where $R_{3n}(h_0, z)$ has a root at $\frac{1}{2}(11+5\sqrt{5})$ and, correspondingly, $R(h_0', z)$ has a root at $\frac{1}{2}(11-5\sqrt{5})$. As n increases it appears that the discriminant $\Delta_{3n}(h)$ has more than two roots which correspond to the roots of (1). As a concrete example, the $6\times\infty$ strip has an irreducible polynomial factor

$$\Lambda^5-\Lambda^4(z^2+4z+1)-2\Lambda^3 z(2z^2+4z+1) \\ +2\Lambda^2 z^3(3z^2+7z+4)+4\Lambda z^5(z-1)-8z^8=0. \tag{5}$$

The resolvent (3) is a menacingly long polynomial

$$R_6(h,z)=2^{10}z^{26}(h-1)^5\sum_{j=0}^{16}\psi_j(h)z^j \qquad (\psi_j(h)=h^{20}\psi_j(h^{-1})) \tag{6}$$

the polynomials ψ_j are listed in the appendix. A portion of the curves $\{z_j(h)\}_6$ is shown in figure 1. That part marked C_6^{1+} corresponds to $|h|=1$ and terminates at the branch point ● ($h=1$). On extending into the domain of real h, C_6^{1+} is continued (represented by the full curve) down to the real axis point denoted by ○. Further exploration of

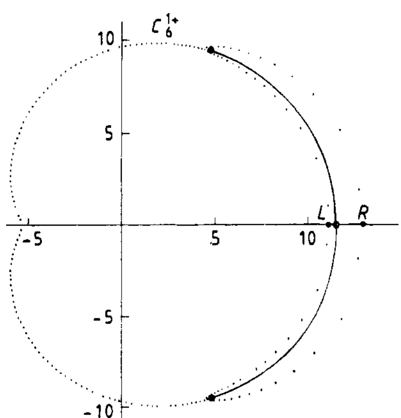

Figure 1. A portion of the algebraic curve $z(h)$ obtained from (6). C_6^{1+} terminates at ● where $h=1$. The banana encasement and the extension of C_6^{1+} are part of the algebraic curve for h real. The three points L, ○ and R correspond to branch points of $z(h)$. L occurs at the exact critical point $z=\frac{1}{2}(11+5\sqrt{5})$.

real h reveals a 'banana'-like encasement of this extension with real axis intersection points denoted by L and R. Both these points and ○ correspond to branch points of the algebraic function $z(h)$ defined by (6), and L is the exact critical point $z_c = \frac{1}{2}(11+5\sqrt{5})$ (numerical identification). Given that L is the positive root of (1) and that (6) is a polynomial with integer coefficients of $z^\alpha h^\beta$ (rational coefficients would suffice) then it is possible to construct an algebraic proof that the negative root of (1) must also correspond to a branch point of the algebraic function $z(h)$ (Burgess 1988). This proof is fairly lengthy and is omitted in the present letter. Armed with this theorem, on further exploration we indeed find this effect. A branch point exists at $h = 0.009\,572\,577\ldots$ corresponding to $z = \frac{1}{2}(11-5\sqrt{5})$. In figure 2 we show the algebraic curves in the domain $0 < h < 0.5$. The sharp protrusion extending from the points P and P^* interesects the real negative axis at the negative root of (1). Having found both points supports that conclusion that they are indeed the algebraic roots of (1) and not simply very close (ten significant figures) approximations.

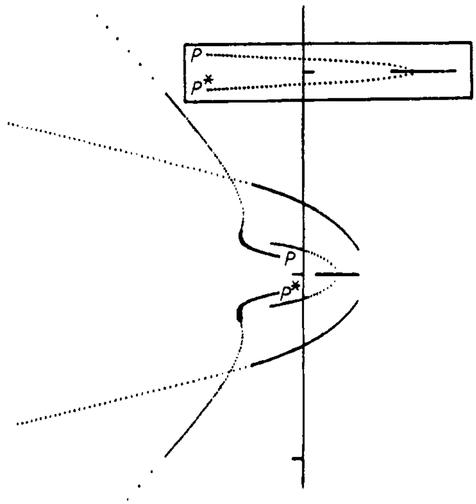

Figure 2. The algebraic curve $z(h)$ obtained from (6) in the range $0 < h < 0.5$. The extension of the curve through P and P^* shown in the insert intersects the real negative axis at $z = \frac{1}{2}(11-5\sqrt{5})$.

Numerically it appears that the 'banana' encasements shown in figure 1 persist for $3n \times \infty$ strips, becoming smaller as the branch point ● approaches the critical point; the intersection point L invariantly occurs at $z = \frac{1}{2}(11+5\sqrt{5})$. Consequently both roots of (1) correspond to branch points of $R_{3n}(z, h) = 0$ and are invariants associated with the semi-infinite lattice sequence $3n \times \infty$.

We are very grateful to Professor D A Burgess for the algebraic proof referred to in the text, and one of us (RWT) is grateful to the SERC for the award of a Research Assistantship.

Appendix

The symmetric polynomials $\psi_j(h)$ in (6) are listed below.

$$\psi_{16} = h^4(h+1)^4(h^2-2)^2(2h^2-1)^2$$

$$\psi_{15} = -h^2(h+1)^2(24h^{14}+56h^{13}-120h^{12}-348h^{11}$$
$$+256h^{10}+974h^9-174h^8-1393h^7-174h^6+\ldots)$$

$$\psi_{14} = (h+1)^2(32h^{18}+80h^{17}-232h^{16}-640h^{15}+1528h^{14}+3992h^{13}-3488h^{12}$$
$$-11\,408h^{11}+2449h^{10}+16\,527h^9+2449h^8-\ldots)$$

$$\psi_{13} = -h(h+1)^2(48h^{16}+1920h^{15}+3752h^{14}-9944h^{13}-22\,816h^{12}+29\,570h^{11}$$
$$+79\,534h^{10}-19\,674h^9-117\,275h^8-19\,674h^7+\ldots)$$

$$\psi_{12} = h(48h^{18}-2464h^{17}-7376h^{16}+42\,772h^{15}+190\,624h^{14}+112\,910h^{13}$$
$$-476\,618h^{12}-644\,656h^{11}+490\,059h^{10}+1376\,400h^9+490\,059h^8-\ldots)$$

$$\psi_{11} = 2h(8h^{18}-812h^{17}-636h^{16}+47\,060h^{15}+165\,988h^{14}-36\,015h^{13}-896\,650h^{12}$$
$$-1117\,583h^{11}+553\,852h^{10}+1859\,374h^9+553\,852h^8-\ldots)$$

$$\psi_{10} = -2h^2(236h^{16}-1084h^{15}-44\,042h^{14}-127\,376h^{13}+355\,917h^{12}+1935\,999h^{11}$$
$$+2222\,328h^{10}-1191\,542h^9-3838\,179h^8-1191\,542h^7+\ldots)$$

$$\psi_9 = -2h^2(24h^{16}-784h^{15}-22\,100h^{14}-36\,740h^{13}+568\,632h^{12}+2373\,405h^{11}$$
$$+2422\,098h^{10}-2497\,638h^9-6266\,082h^8-2497\,638h^7+\ldots)$$

$$\psi_8 = h^3(424h^{14}+12\,184h^{13}-17\,156h^{12}-905\,628h^{11}-3574\,282h^{10}-3047\,939h^9$$
$$+6860\,646h^8+14\,454\,068h^7+6860\,646h^6-\ldots)$$

$$\psi_7 = h^3(40h^{14}+1632h^{13}-18\,972h^{12}-424\,298h^{11}-1758\,430h^{10}-1147\,168h^9$$
$$+5823\,137h^8+11\,288\,510h^7+5823\,137h^6-\ldots)$$

$$\psi_6 = h^4(44h^{12}-5696h^{11}-123\,338h^{10}-588\,228h^9-269\,981h^8$$
$$+3167\,565h^7+6007\,034h^6+3167\,565h^5-\ldots)$$

$$\psi_5 = -h^4(8h^{12}+816h^{11}+22\,210h^{10}+135\,416h^9+50\,216h^8$$
$$-1130\,775h^7-2203\,410h^6-1130\,775h^5+\ldots)$$

$$\psi_4 = -h^5(52h^{10}+2282h^9+20\,856h^8+13\,010h^7$$
$$-261\,519h^6-557\,086h^5-261\,519h^4+\ldots)$$

$$\psi_3 = -4h^6(27h^8+498h^7+909h^6-9160h^5-23\,900h^4-9160h^3+\ldots)$$

$$\psi_2 = -2h^7(45h^6+278h^5-1307h^4-5404h^3-1307h^2+\ldots)$$

$$\psi_1 = -4h^8(8h^4-9h^3-186h^2-9h+8)$$

$$\psi_0 = -4h^9(h^2-6h+1).$$

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